2. Essential /Elementary Functions

Algebraic and Transcendental Functions – "Essential Functions"

2.1 Algebraic Functions

Algebraic Functions are functions which can be constructed by using algebraic operations on the independent variable x that is: addition, subtraction, multiplication, division & taking roots.

The most important algebraic functions are :

1. *Polynomial Functions.*
2. *Power Functions.*
3. *Rational Functions.*
4. *Irrational Functions.*

2.2 Transcendental Functions

Transcendental Functions are non-algebraic functions, i.e. they cannot be constructed by performing the algebraic operations on the independent variable x. It has a special definition.

The most frequently used of them are:

1. *Trigonometric Functions.*
2. *Inverse Trigonometric Functions.*
3. *Exponential Functions.*
4. *Logarithmic Functions.*
5. *Hyperbolic Functions.*
6. *Inverse Hyperbolic Functions.*

2.2.1 Polynomial Functions:

Definition: A function $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ is called a ***polynomial function*** of degree n, where n is a nonnegative integer real number, and the coefficients $a_n, a_{n-1}, ..., a_1$, a_0 are real numbers (constants) called the ***coefficients*** of the polynomial function, with $a_n \neq 0$. *The domain of any polynomial function is* $R=(-\infty,\infty)$.

e.g. $P_5(x)=3x^5+2x^4+7x^2+\sqrt{10}$, is a polynomial function *of degree 5*.

*** Degree 0:** $P_0(x)=a$ *Constant function*

The graph of this function is the straight line: $y=c$, which is parallel to the x-axis.

*** Degree 1:** $f(x)=ax+b \equiv P_1(x)$ *Linear function*

The graph of this function is the straight line: $y=mx+c$

*** Degree 2:** $f(x)=ax^2+bx+c \equiv P_2(x)$ *Quadratic function*

Its graph is always a *"Parabola"* obtained by shifting the parabola: $y=ax^2$, the parabola *Opens upwards if* $a>0$ *and downwards if* $a<0$.

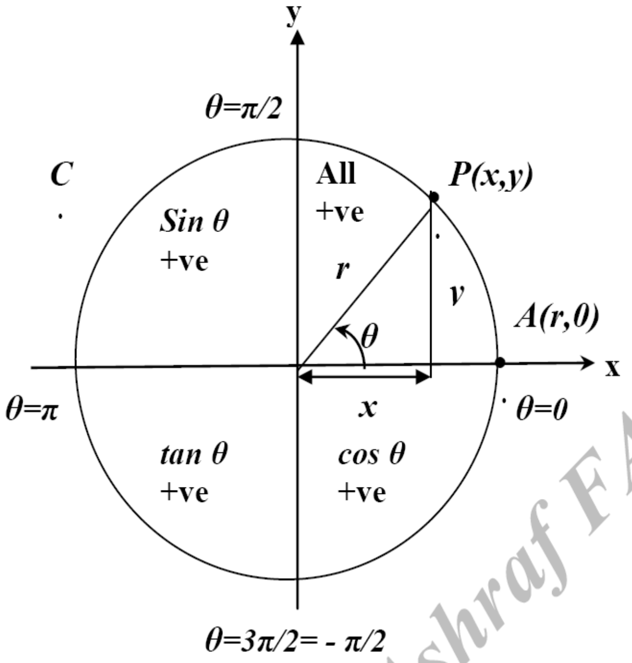
(a) $y=x^2+x+1$ (b) $y=-2x^2+3x+1$

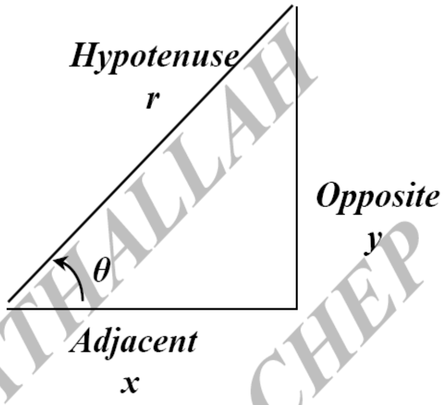
Theorems:

1. *A Polynomial of degree* n *has at most* n *distinct zeros (roots).*
2. *For any Polynomial* f , $f(a)=0$ *if and only if* $(x-a)$ *is a factor of* $f(x)$.

2.3 Transcendental Functions:

2.3.1 Trigonometric Functions





The ***Six Trigonometric Functions*** :

sine, cosine, tangent, cotangent, secant, and cosecant are defined and abbreviated as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{1}{\csc\theta} = -\sin(-\theta)\ ;\ \text{Odd } f^{\underline{n's}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} = \frac{1}{\sec\theta} = \cos(-\theta)\ ;\text{Even } f^{\underline{n's}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta} = -\tan(-\theta)$$

Remark:

In all applications, angles are usually measured in ***radians*** unless otherwise specified. For example: *sin 30* *means that sine of 30 radians whereas the sine of 30 degree is written as* $\sin 30^{o}$.

$$2\pi\ rad = 360^{o} \Rightarrow 1\,radian = \frac{180^{o}}{\pi}\ \&\ 1^{o} = \pi/180.$$

B. Some Important Trigonometric Identities:

Equations that describe relationships between trigonometric functions are called ***Fundamental Trigonometric Identities.***

1. The Principle Identity: (also called "Pythagorean Identities)

$$\boxed{\sin^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad \begin{cases} \tan^2\theta + 1 = \sec^2\theta \\ 1 + \cot^2\theta = \csc^2\theta \end{cases}}$$

2. $$\boxed{\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \ \ \& \ \ \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta.}$$

3. The Sum formulas:

$$\begin{aligned} \sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b, \\ \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b, \\ \tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \end{aligned}$$

4. The double-angle formulas:

$$\begin{aligned} &\sin 2\theta = 2\sin\theta\cos\theta, \\ &\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \end{aligned}$$

5. The half-angle formulas:

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \ \& \ \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta).$$

6. The Multiplication formulas:

$$\begin{aligned} \sin a \cos b &= \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right], \\ \sin a \sin b &= \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right], \\ \cos a \cos b &= \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]. \end{aligned}$$

C. Solving Equations Involving Trigonometric Functions:

Example 2: *Solve the following equation for x,* $\cos 4x = 0$

Solution From the properties of the cosine function, we have

$$4x = (2n+1)\frac{\pi}{2} \quad \Rightarrow x = (2n+1)\frac{\pi}{8} \ ; \ n = 0, \pm 1, \pm 2, .. \qquad \blacksquare$$

Example 3:

$\sin(\theta) = \frac{1}{2}$ if

i = $\theta \in [0, 2\pi]$

ii = $\theta \in \mathbb{R}$

Solution:-

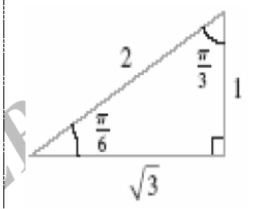
i- $\theta \in [0, 2\pi]$

$$\theta = \pi/6, \quad \theta = (5\pi/6)$$

ii- $\theta \in \mathbb{R}$

$$\theta = \pi/6 + 2n\pi, \quad n = 0, \pm 1, \quad \pm 2.....$$

$$\theta = (5\pi/6) + 2n\pi, \qquad n = 0, \quad \pm 1, \quad \pm 2,.....$$



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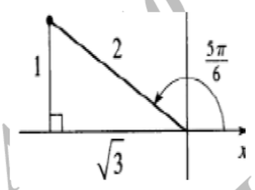


Example 4:

Solve $Sec^2\theta - (1+\sqrt{3})\ Tan\theta + (\sqrt{3}-1) = 0, \theta \in [0, 2\pi]$

Solution:-

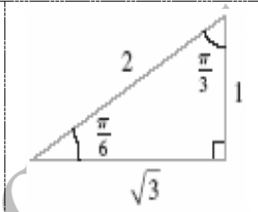
$$(1+Tan^2\theta) - (1+\sqrt{3})Tan\theta + (\sqrt{3}-1) = 0$$

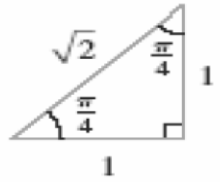
$$Tan^2\theta - (1+\sqrt{3})Tan\theta + \sqrt{3} = 0$$

$$(Tan\theta - 1)(Tan\theta - \sqrt{3}) = 0$$

$$Tan\theta = 1, \ Tan\theta = \sqrt{3}$$

$$\theta = \pi/4, \pi/3$$





Example 5:

Solve $Cos(x) + Cos(2x) = 0 \qquad x \in [0, 2\pi]$

Solution:-

$$2Cos^2(x) + Cos(x) - 1 = 0$$

$$(2Cos(x) - 1)(Cos(x) + 1) = 0$$

$$Cos(x) = 1/2, -1$$

$$x = \pi/3, \pi, 5\pi/3\ , \pi$$

EXAMPLE 4.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations (a) $2\sin x - 1 = 0$ and (b) $\cos^2 x - 3\cos x + 2 = 0$.

Solution For (a), notice that $2\sin x - 1 = 0$ if $2\sin x = 1$ or $\sin x = \frac{1}{2}$. From the unit circle, we find that $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Since $\sin x$ has period 2π, additional solutions are $\frac{\pi}{6} + 2\pi$, $\frac{5\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$ and so on. A convenient way of indicating that *any* integer multiple of 2π can be added to either solution is to write $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, for any integer n. Part (b) may look rather difficult at first. However, notice that it looks like a quadratic equation using $\cos x$ instead of x. With this clue, you can factor the left-hand side to get

$$0 = \cos^2 x - 3\cos x + 2 = (\cos x - 1)(\cos x - 2),$$

from which it follows that either $\cos x = 1$ or $\cos x = 2$. Since $-1 \le \cos x \le 1$ for all x, the equation $\cos x = 2$ has no solution. However, we get $\cos x = 1$ if $x = 0, 2\pi$ or any integer multiple of 2π. We can summarize all the solutions by writing $x = 2n\pi$, for any integer n. ■

2.4 Exponentials & Logarithms Functions: (Will be studied in Ch. 7.)

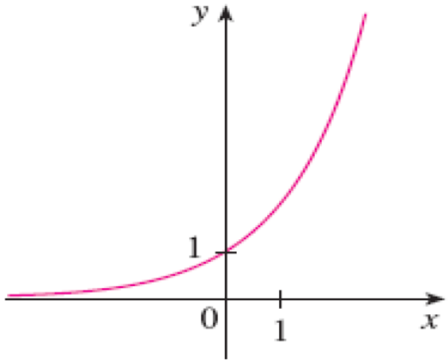
2.4.1 Exponential Functions (Stewart P. 33)

Let $a > 0$ be any real number. The function $f(x)= a^x$ is called the "exponential function" with base a and exponent x. Its domain is $(-\infty,\infty)$ and its range is $(0,\infty)$.

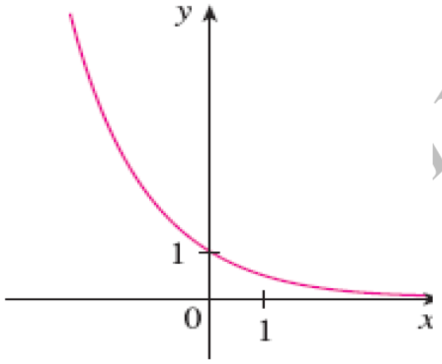
Example 1

Sketch and compare the graphs of $y = 2^x$ and $y = (1/2)^x$.

Solution We make a table of values, carefully plot the points, and draw the curves:



(a) $y = 2^x$



(b) $y = (0.5)^x$

* **Some basic properties of exponential functions:**

a) $a^x.a^y = a^{x+y}$ (b) $\left(a^x\right)^n = a^{nx}$ (c) $a^{-x} = \dfrac{1}{a^x}$ (d) $\dfrac{a^x}{a^y} = a^{x-y}$

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The Natural Exponential Function $f(x) = e^x$:

The Natural Number "e":

In the honor of the Swiss mathematician scientist ***Leonhard Euler*** (1707-1783), the number ***"e"*** is defined by: $e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$. In fact, the ***Natural Number "e" is an irrational number which has an approximated value up to ten digits, given by:***

$$\boxed{e \approx 2.718\,281\,828\,4}$$

2.4.2 Logarithmic Functions (Stewart P. 34))

The ***General & Natural Logarithmic Functions*** are considered as the ***Inverse Functions of the General & Natural Exponential Functions,***

$$f(x)= \log_a x \ \&\ f(x)= \log_e x \equiv \ln x \ ,$$

respectively. The inverse function of the exponential function $f(x) = a^x$ is called the ***logarithmic function*** to the base a and is denoted by $\log_a x$ .Therefore,

$$\log_a x = y \Leftrightarrow a^y = x .$$

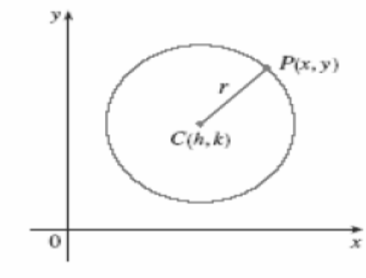
The number a is called ***the base of the logarithm***. The following example indicates the same statements for both exponential and logarithmic forms.

III GRAPHS OF SECOND-DEGREE EQUATION

1. Circles

Equation of a circle with centre (h, k) and radius r is

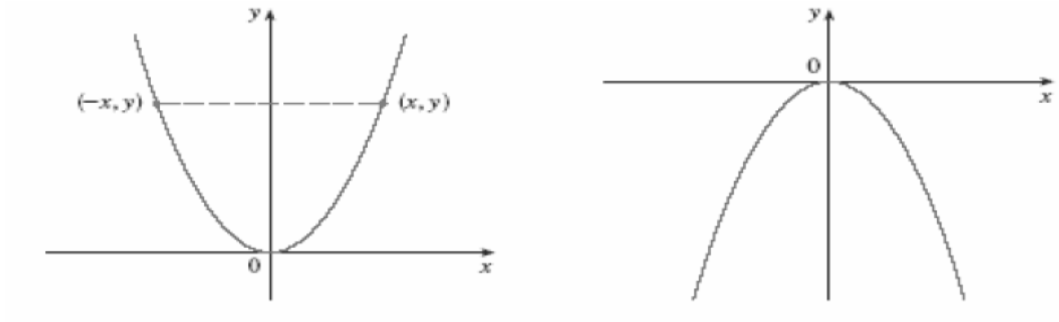
$$(x-h)^2+(y-k)^2=r^2$$



2. Parabolas

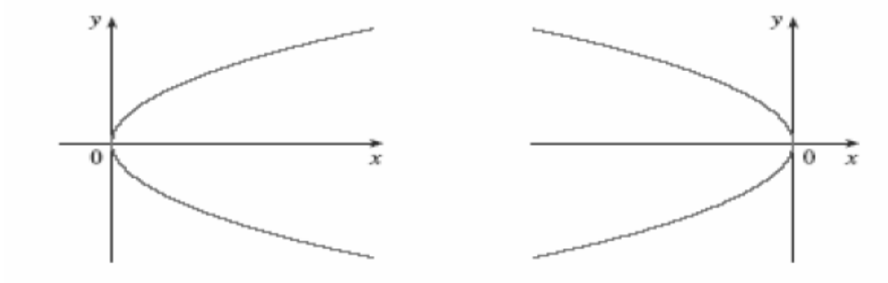
Equation of a Parabola is $y = ax^2 + bx + c$

e.g:



(a) $y = ax^2,\ a > 0$ (b) $y = ax^2,\ a < 0$

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$

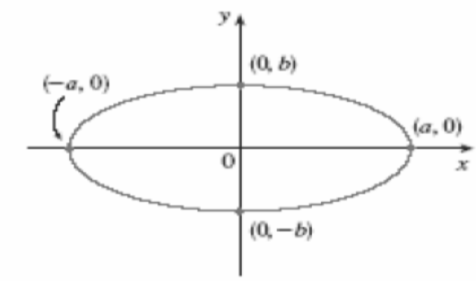


(a) $x = ay^2,\ a > 0$ (b) $x = ay^2,\ a < 0$

3. Ellipses

The curve with equation:

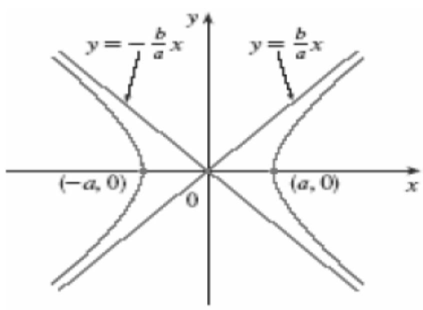
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is called an ellipse.



4. Hyperbolas

The curve with equation:

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is called a hyperbola.



Quadric Surfaces and Conic Sections

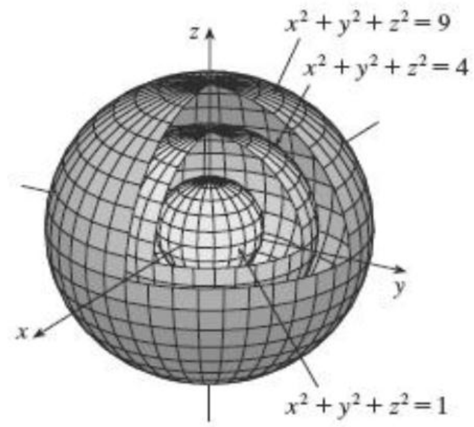
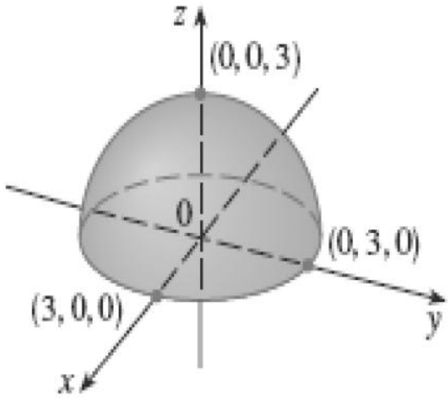


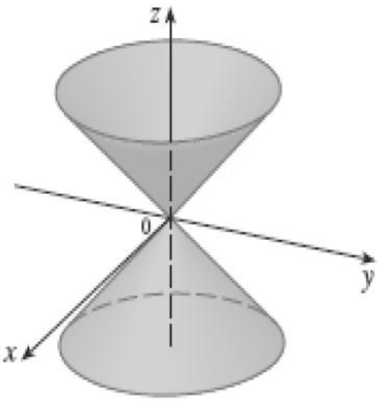
FIGURE 20

Sphere

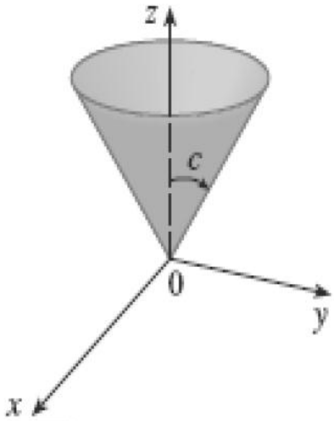


$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

Hemisphere



Cone



Half-Cone